Coordinate Geometry

1. The diagram shows a trapezium ABCD such that BC is parallel to AD and perpendicular to CD.
   i) Find the coordinates of vertex D
   ii) Point E lies on BC such that the area of triangle ACE is \( \frac{1}{2} \) of the area of triangle ABE. Find the coordinates of E.
   iii) Point F lies on AD produce such that it forms a parallelogram with vertices A, B and C. Find the possible coordinates of F.
   iv) Determine the ratio of the area of triangle ACB to the parallelogram AFBC.

i) Gradient of AD = Gradient of BC = \( \frac{5-(-4)}{-1-2} = -3 \)

Gradient of CD = \( \frac{-1}{-3} = \frac{1}{3} \)

Equation of AD: \( y - 0 = -3(x - 4) \)
\( y = -3x + 12 \) \( - - - (1) \)

Equation of CD: \( y - (-4) = \frac{1}{3}(x - 2) \)
\( y = \frac{1}{3}x - \frac{14}{3} \)
\( 3y = x - 14 \) \( - - - (2) \)

Sub (1) into (2):
\( 3(-3x + 12) = x - 14 \)
\( -9x + 36 - x + 14 = 0 \)
\( x = 5 \)
\( y = -3 \)

ii) \( \frac{\text{area of } ACE}{\text{area of } ABE} = \frac{1}{2} \)

\( \frac{1}{2} \times EC \times \text{Height} = \frac{1}{2} \)

\( \frac{1}{2} \times EB \times \text{Height} = \frac{1}{2} \)

\( \frac{EC}{EB} = \frac{1}{2} \)

By Ratio Theorem, Coordinates of E = \( \left( \frac{1 \times -1 + 2 \times 2}{2+1}, \frac{1 \times 5 + 2 \times -4}{2+1} \right) = (-1, -1) \)

iii) Case 1: For parallelogram arranged as AFBC
\[ \overrightarrow{BC} = \overrightarrow{FA} = \left( \frac{3}{-9} \right) \]
\[ \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \left( \frac{4}{0} \right) + \left( \frac{-3}{9} \right) = \left( \frac{1}{9} \right) \]
Coordinates of \( F \) is \( (1,9) \)

Case 2: For parallelogram arranged as ABFC
\[ \overrightarrow{BC} = \overrightarrow{AF} = \left( \frac{3}{-9} \right) \]
\[ \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \left( \frac{4}{0} \right) + \left( \frac{3}{-9} \right) = \left( \frac{7}{-9} \right) \]
Coordinates of \( F \) is \( (7,-9) \)
\[ \therefore \text{possible coordinates of } F \text{ are } (1,9) \text{and} (7,-9) \]

iv) \[ \frac{\text{area of } \triangle ABC}{\text{area of } \triangle AFBC} = \frac{\frac{1}{2} \times BC \times \text{height}}{BC \times \text{height}} = \frac{1}{2} \]
2 Point A has coordinates (2,3) and line \( l_1 \) has equation \( 2y = 4x + 5 \).

a) Find the coordinates of the foot of the perpendicular from Point A to line \( l_1 \).

b) Find the shortest distance from Point A to line \( l_1 \).

c) Point B is the reflection of Point A on the line \( l_1 \), find the coordinates of B.

a) \textit{Let the foot of the perpendicular from Point A to line } \( l_1 \) \textit{be Point C}

\text{Gradient of } \( l_1 \) \text{ is } \frac{4}{2} = 2

\text{Gradient of } \perp \text{ line } = -\frac{1}{2}

\text{Equation of } \perp \text{ line: } (y - 3) = -\frac{1}{2}(x - 2)

\text{Equating } l_1 \text{ and } \perp \text{ line: } 2x + \frac{5}{2} = -\frac{1}{2}x + 4

4x + 5 + x - 8 = 0

5x = 3

x = 0.6

y = 3.7

\text{Coordinates of } C \text{ is } (0.6, 3.7)

b) \text{ Length of } AC = \sqrt{(2 - 0.6)^2 + (3 - 3.7)^2}

= 1.57 \text{ units (3 s.f.)}

c) \( C \) \text{ is the mid-point of } AB

\frac{2 + x}{2} = 0.6

\frac{x}{2} = -0.8

\frac{3 + y}{2} = 3.7

y = 4.4

\text{Coordinates of } B \text{ is } (-0.8, 4.4)
3 The equation of the perpendicular bisector of the line segment which joins $A(2,3)$ and $B(h,k)$ is $y = x - 1$. Find the value of $h$ and of $k$.

Gradient of perpendicular bisector = 1

Gradient of $AB = -1 = \frac{(k-3)}{(h-2)}$

\[ k - 3 = -h + 2 \]
\[ k = -h + 5 \quad \text{(1)} \]

Midpoint of $AB = \left( \frac{2+h, 3+k}{2} \right)$

\[ \frac{3+k}{2} = \frac{2+h}{2} - 1 \]
\[ 3 + k = 2 + h - 2 \]
\[ k = h - 3 \quad \text{(2)} \]

Sub (1) to (2): $-h + 5 = h - 3$

\[ 2h = 8 \]
\[ h = 4 \]
\[ k = 1 \]

4 The diagram shows 3 vertices of a parallelogram. Given $A(1,2)$, $B(3,0)$ and $O$, find the possible positions of the fourth vertex.

Ans: $(4,2), (-2,2), (2,-2)$
5 The diagram above (not drawn to scale) shows kite \(ABCD\) with \(DC\) parallel to the \(x\)-axis. The area of triangle \(ADC\) is 3 times that of triangle \(ABC\). Given that \(C(7, -2)\) and the equation of the diagonal \(BD\) is \(2y = x\), find

i) Coordinates of \(D\)

Sub \(y = -2\) into \(2y = x\):
\[x = -4\]
Coordinates of \(D\) is \((-2, -4)\)

ii) Gradient of \(DB\) = \(\frac{1}{2}\)

Gradient of \(AC\) = \(-2\)

Equation of \(AC\): \((y - (-2)) = -2(x - 7)\)
\[y = -2x + 12\]
Equate \(AC\) and \(BD\): \(-2x + 12 = \frac{1}{2}x\)
24 = 5x
\[x = 4.8\]
\[y = 2.4\]
Coordinates of \(E\) is \((4.8, 2.4)\)

iii) \(E\) is the midpoint of \(AC\).
\[
\left(\frac{x+7}{2}, \frac{y-2}{2}\right) = (4.8, 2.4)
\]
\[x = 2.6\]
\[y = 6.8\]
Coordinates of \(A\) is \((2.6, 6.8)\)

iv) Given that \[\frac{\text{Area } ADC}{\text{Area } ABC} = \frac{3}{1}\]
\[\frac{DE}{EB} = \frac{3}{1}\]

Using Ratio Theorem:
\[\left(\frac{1\times(-2)+3\times(x)}{3+1}, \frac{1\times(-4)+3\times(y)}{3+1}\right) = (4.8, 2.4)\]
\[x = 7.07\]
\[y = 4.53\]
Coordinates of \(B\) is \((7.07, 4.53)\)
6 Three points A, B and C lies on a straight line such that \( AB = 2BC \). The coordinates of point B is 
\((4, -2)\) and \( \tan \theta = \frac{2}{3} \). Find the

i) equation of line \( AC \)

ii) coordinates of A and C

iii) coordinates of the point on line \( AC \) that is closest to \( O \).

(Leave your answer to the nearest 3 s.f.)

i) Since \( \tan \theta = \frac{2}{3} \),
Gradient of \( AC \) = \( \frac{2}{3} \)

Equation of \( AC \): \( y - (-2) = \frac{2}{3} (x - 4) \)
\[ y = \frac{2}{3} x - \frac{8}{3} - 2 \]
\[ 3y = 2x - 14 \]  
\[-(1)\]

ii) C is on the x-axis, \( y = 0, \)
\[ 0 = 2x - 14 \]
\[ x = 7 \]
\[ \therefore C(7, 0) \]

\( AB = 2BC \)
\[ \sqrt{(x - 4)^2 + (y + 2)^2} = 2\sqrt{(4 - 7)^2 + (-2 - 0)^2} \]
\[ (x - 4)^2 + (y + 2)^2 = 4(9 + 4) \]
\[ (x - 4)^2 + (y + 2)^2 = 52 \]  
\[-(2)\]

Sub (1) into (2):
\[ (x - 4)^2 + \left( \frac{2x - 14}{3} + 2 \right)^2 = 52 \]
\[ (x - 4)^2 + \left( \frac{2}{3} x - \frac{8}{3} \right)^2 = 52 \]
\[ x^2 + 16 - 8x + \frac{4}{9} x^2 + \frac{64}{9} - \frac{32}{9} x - 52 = 0 \]
\[ 13x^2 - 104x - 260 = 0 \]
\[ (x + 2)(x - 10) = 0 \]
\[ x = -2 \] or \( x = 10 \) (Ref)
\[ y = -6 \]
\[ \therefore A(-2, -6) \]

iii) Let the point on AC that is closest to \( O \) be \( D \)

Gradient of \( OD \) = \( \frac{-1}{2} = -\frac{3}{2} \)

Equation of \( OD \): \( y = -\frac{3}{2} x \)  
\[-(3)\]

Sub (1) with (3):
\[ 3 \left( -\frac{3}{2} x \right) = 2x - 14 \]
\[ -9x = 4x - 28 \]
\[ x = \frac{28}{13} = 2.15 \text{ (3 s.f.)} \]
\[ y = \frac{-42}{13} = -3.23 \text{ (3 s.f.)} \]

Coordinates of point is \((2.15, -3.23)\)
The diagram shows a trapezium OABC. The equation of \( OA \) is \( y = x \) and the equation of \( OC \) is \( 2y + x = 0 \). Line \( OA \) is parallel to \( CB \) and perpendicular to \( AB \). Point \( B \) is on the \( x \)-axis. The length of \( OA \) is \( 4\sqrt{2} \) units.

i) Find the coordinates of \( A \)
ii) Find the coordinates of \( B \)
iii) Find the coordinates of \( C \).
iv) Hence, calculate the area of trapezium \( OABC \).

i) Let the coordinates of \( A \) be \((x, x)\)

\[ OA = \sqrt{(x - 0)^2 + (x - 0)^2} \]
\[ 4\sqrt{2} = \sqrt{2x^2} \]
\[ 32 = 2x^2 \]
\[ x = 4 \quad \text{or} \quad x = -4 \ (R e f) \]

Coordinate of \( A \) is \((4, 4)\)

ii) Gradient of \( OA = 1 \)
Gradient of \( AB = -1 \)
Equation of \( AB \): \((y - 4) = -1(x - 4)\)
y = \(-x + 8\)
When \( y = 0 \): 0 = \(-x + 8\)
x = 8
Coordinates of \( B \) is \((8, 0)\)

iii) Gradient of \( CB = \) Gradient of \( OA = 1 \)
Equation of \( CB \): \((y - 0) = 1(x - 8)\)
y = \(x - 8\)
Equate equation \( OC \) with equation \( CB \):
\[-\frac{x}{2} = x - 8\]
\[-x = 2x - 16\]
x = \(\frac{16}{3} = 5\frac{1}{3}\)
y = \(-\frac{8}{3} = -2\frac{2}{3}\)
Coordinate of \( C \) is \((5\frac{1}{3}, -2\frac{2}{3})\)

iv) Area \( OABC = \frac{1}{2} \left|\begin{array}{cccc}
4 & 0 & 5\frac{1}{3} & 8 & 4 \\
4 & 0 & -2\frac{2}{3} & 0 & 4 \\
\end{array}\right| \\
= \frac{1}{2} \left( 4 \times 0 + 0 \times -2\frac{2}{3} + 5\frac{1}{3} \times 0 + 8 \times 4 \right) - \left( 4 \times 0 + 0 \times 5\frac{1}{3} - 2\frac{2}{3} \times 8 + 0 \times 4 \right) \right) \\
= \frac{1}{2} \left( 32 + 6\frac{4}{3} \right) \\
= 26\frac{2}{3} \ units^2
8. ABCD is a trapezium with AB parallel to BC. The equation of DC is $6y = 11x - 41$. Given that midpoint of AD lies on the y-axis and the midpoint of BD lies on the x-axis, find

i) the coordinates of $D$
ii) the coordinates of $C$
iii) area of ABCD
iv) the perpendicular distance between AD and BC (leaving your answer to 3 s.f.)

i) Let the coordinates of $D$ be $(x, y)$

Since the midpoint of $AD$ lies on y-axis,

\[
\frac{x - 1}{2} = 0
\]

\[x = 1\]

Since the midpoint of BD lies on the x-axis,

\[
\frac{y + 5}{2} = 0
\]

\[y = -5\]

The coordinates of $D$ is $(1, -5)$

ii) Gradient of $AD = \frac{4 - (-5)}{-1 - 1} = -\frac{9}{2}$

Gradient of $BC = \text{Gradient of } AD = -\frac{9}{2}$

Equation of BC: $(y - 5) = -\frac{9}{2}(x - 3)$

$y = -\frac{9}{2}x + 18.5$

Equate BC and DC:

\[-\frac{9}{2}x + 18.5 = \frac{11}{6}x - \frac{41}{6}\]

\[x = 4\]

\[y = 0.5\]

The coordinates of $C$ is $(4, 0.5)$

iii) Area of ABCD = \[
\frac{1}{2} \left| -1 \quad 4 \quad -5 \quad 0.5 \quad 5 \quad 4 \right| \]

= \[
\frac{1}{2} \left( (-1 \times -5 + 1 \times 0.5 + 4 \times 5 + 3 \times 4) - (4 \times 1 - 5 \times 4 + 0.5 \times 3 + 5 \times 1) \right) \]

= \[
\frac{1}{2} \times 57 \]

= 28.5 units$^2$

iv) Length of AD = \[
\sqrt{(4 - (-5))^2 + ((-1) - 1)^2} = \sqrt{85}
\]

Length of BC = \[
\sqrt{(5 - 0.5)^2 + (4 - 3)^2} = \sqrt{21.25}
\]

Area of ABCD = \[
\frac{AD + BC}{2} \times \text{Height}
\]

\[28.5 = \frac{\sqrt{85} + \sqrt{21.25}}{2} \times \text{Height} \]

Height = 4.12 units
In the diagram, ABCD is a rectangle. The coordinates of A are (-1,2) and the equation of BC is \(3y + x = 25\). Given that the area of ABCD is 80 units\(^2\), find the coordinates of B, C and D.

Equation of BC: \(y = -\frac{x}{3} + \frac{25}{3}\)
Gradient of BC = \(-\frac{1}{3}\)
Gradient of AB = 3
Equation of AB: \((y - 2) = 3(x + 1)\)
\(y = 3x + 5\)
Equate equations of AB and BC:
\(3x + 5 = -\frac{x}{3} + \frac{25}{3}\)
\(x = 1\)
\(y = 8\)
Coordinates of B are (1,8)

Length of AB = \(\sqrt{(8 - 2)^2 + (1 - (-1))^2} = \sqrt{40}\)
Length of BC = \(\frac{80}{\sqrt{40}} = 4\sqrt{10}\)
Let the coordinates of C be \((x, y)\)
Length of BC = \(\sqrt{(8 - y)^2 + (1 - x)^2} = 4\sqrt{10}\)
\((8 - y)^2 + (1 - x)^2 = 160\) --- (2)
Substitute Equation (1) with (2):
\(\left(8 + \frac{x}{3} - \frac{25}{3}\right)^2 + (1 - x)^2 = 160\)
Solve for \(x\):
\(x = 13\) or \(x = -11\) (Rej)
\(y = 4\)
Coordinates of C are (13,4)
Equation of CD: \((y - 4) = 3(x - 13)\)
\(y = 3x - 35\)
Equation of AD: \((y - 2) = -\frac{1}{3}(x + 1)\)
\(y = -\frac{1}{3}x + \frac{5}{3}\)
Equate Equations of CD with AD:
\(3x - 35 = -\frac{1}{3}x + \frac{5}{3}\)
\(x = 11\)
\(y = -3\)
Coordinates of D are (11, -3)
The diagram shows a rhombus ABCD. Two of the points are A(1,-1) and C(7,5). Point D lies on the y-axis.

i) Find the coordinates of D

ii) Find the coordinates of B

iii) Find the area of rhombus ABCD

iv) Calculate the perpendicular distance from C to AB.

i) Mid point of AC = \( \left( \frac{7+1}{2}, \frac{5-1}{2} \right) = (4, 2) \)

Gradient of AC = \( \frac{5 - (-1)}{7 - 1} = \frac{1}{1} = 1 \)

Gradient of perpendicular bisector of AC = \( -\frac{1}{1} = -1 \)

Equation of DB: \((y - 2) = -1(x - 4)\)

\( y = -x + 6 \)

When \( x = 0 \), \( y = 0 + 6 = 6 \)

Coordinates of D are \((0, 6)\)

ii) Let the coordinates of B be \((x, y)\),

Mid point of DB = Mid point of AC

\( \left( \frac{0+x}{2}, \frac{6+y}{2} \right) = (4, 2) \)

\( \frac{0+x}{2} = 4 \) and \( \frac{6+y}{2} = 2 \)

\( x = 8 \) and \( y = -2 \)

Coordinates of B are \((8, -2)\)

iii) Area of ABCD = \( \frac{1}{2} \left| \begin{array}{cc} 1 & 8 & 7 & 0 & 1 \\ -1 & -2 & 5 & 6 & -1 \end{array} \right| \)

= \( \frac{1}{2} \left( (1 \times -2 + 8 \times 5 + 7 \times 6 + 0 \times -1) - (-1 \times 8 - 2 \times 7 + 5 \times 0 + 6 \times 1) \right) \)

= \( \frac{1}{2} \times 89 \)

= 44.5 units\(^2\)

iv) Length of AB = \( \sqrt{(-2 - (-1))^2 + (8 - 1)^2} = \sqrt{50} \)

Perpendicular distance = \( \frac{44.5}{\sqrt{50}} = 6.29 \) units (3 s. f.)