Polynomials

1. Without using long division, find the remainder when $2x^6 + x^4 - 15x^2 - 14$ is divided by $x^2 + 2$.

Sub $x^2 = y$:

$f(y) = 2y^3 + y^2 - 15y - 14$

Divided by $y + 2$.

By Remainder Theorem,

$f(-2) = 2(-2)^3 + (-2)^2 - 15(-2) - 14$

$= 4$

∴ The remainder 4.

2. A cubic polynomial, $f(x)$, leaves a remainder of 12 when divided by $x$ and $f(x + 1) - f(x - 1) \equiv 12x^2 - 12x - 42$. By substituting suitable values of $x$,

a) Find the remainder when $f(x)$ is divided by $(x - 2)$

b) Show that $f(-2) = 30$

c) Show that $(x - 4)$ is a factor of $f(x)$.

a) Since $f(x)$ leaves a remainder of 12 when divided by $x$,

$f(0) = 12$

Sub $x = 1$:

$f(1 + 1) - f(1 - 1) = 12(1)^2 - 12(1) - 42$

$f(2) - f(0) = 12 - 12 - 42$

$f(2) = -30$

The remainder when $f(x)$ is divided by $(x - 2)$ is $-30$.

b) Sub $x = -1$:

$f(-1 + 1) - f(-1 - 1) = 12(-1)^2 - 12(-1) - 42$

$f(0) - f(-2) = 12 + 12 - 42$

$12 - f(-2) = 12 + 12 - 42$

$f(-2) = 30$ (Shown)

c) Sub $x = 3$:

$f(3 + 1) - f(3 - 1) = 12(3)^2 - 12(3) - 42$

$f(4) - f(2) = 108 - 36 - 42$

$f(4) - (-30) = 108 - 36 - 42$

$f(4) = 0$

By Factor theorem, since $f(4) = 0$, $(x - 4)$ is a factor of $f(x)$.
3 Given that \((x - 1)(x - 2)(Ax + B) + C(x - 2) + D = 3x^3 - 7x^2 + 3x + 2\) for all values of \(x\), find \(A, B, C\) and \(D\).

By comparing \(x^3\) coefficients:
\[A = 3\]

Sub \(x = 2\):
\[D = 3(2)^3 - 7(2)^2 + 3(2) + 2 = 4\]

Sub \(x = 1\):
\[-C + D = 3 - 7 + 3 + 2\]
\[C = 3\]

Comparing \(x\)-independent term:
\[2B - 2C + D = 2\]
\[B = 2\]

4 \((x - 2)\) is a factor of \(g(x) + 5\), where \(g(x)\) is a polynomial. Find the remainder when \(f(x) = (2x^3 + 3x^2 - 4)g(x)\) is divided by \((x - 2)\).

By Factor Theorem: \(g(2) + 5 = 0\)
\[g(2) = -5\]
By Remainder Theorem:
Remainder = \(f(2) = (2(2)^3 + 3(2)^2 - 4)g(2)\)
\[= (24)(-5)\]
\[= -120\]
5. The term containing the highest power of \( x \) in the polynomial \( f(x) \) is \( x^4 \) and the roots of \( f(x) = 0 \) are -6 and 3. \( f(x) \) has a remainder of -84 when divided by \( (x - 1) \) and a remainder of -96 when divided by \( (x - 2) \). Find the expression for \( f(x) \).

\[
f(x) = (x - 3)(x + 6)(x + a)(x + b)
\]

\[
f(1) = (1 - 3)(1 + 6)(1 + a)(1 + b)
\]

\[
-84 = (-14)(1 + a)(1 + b)
\]

\[
b = -\frac{84}{(-14)(1+a)} - 1 \quad - (1)
\]

\[
f(2) = (2 - 3)(2 + 6)(2 + a)(2 + b)
\]

\[
-96 = -8(2 + a)(2 + b) \quad - (2)
\]

Sub (1) into (2):

\[
-96 = -8(2 + a)\left(2 - \frac{84}{(-14)(1+a)} - 1\right)
\]

\[
12 = (2 + a)\left(1 + \frac{6}{(1+a)}\right)
\]

\[
12 = (2 + a)\left(\frac{1+a+6}{1+a}\right)
\]

\[
12(1 + a) = (2 + a)(a + 7)
\]

\[
12 + 12a = 2a + a^2 + 7a + 14
\]

\[
a^2 - 3a + 2 = 0
\]

\[
(a - 1)(a - 2) = 0
\]

\[
a = 1 \, or \, a = 2
\]
6 Express \( \frac{3x^2+5}{x^4-1} \) in partial fractions

\( x^4 - 1^4 = (x^2 - 1)(x^2 + 1) \)

\( = (x + 1)(x - 1)(x^2 + 1) \)

\( \frac{3x^2+5}{x^4-1} = \frac{3x^2+5}{(x+1)(x-1)(x^2+1)} \)

Let \( \frac{3x^2+5}{(x+1)(x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} \)

\( 3x^2 + 5 = A(x^2 + 1)(x - 1) + B(x + 1)(x^2 + 1) + (Cx + D)(x + 1)(x - 1) \)

Let \( x = -1, -4A = 8 \)

\( A = -2 \)

Let \( x = 1, 4B = 8 \)

\( B = 2 \)

Let \( x = 0, -A + B - D = 5 \)

\( D = -1 \)

Comparing \( x^3 \) coefficient, \( A + B + C = 0 \)

\( C = 0 \)

\( \therefore \frac{3x^2+5}{x^4-1} = \frac{-2}{x+1} + \frac{2}{x-1} + \frac{-1}{x^2+1} \)
Given that \( f(x) = 4x^3 - 2x^2 + 5x - 1 \), find

i) the remainder when \( f(x) \) is divided by \( (x - 1) \)

ii) the remainder when \( f(x - 8) \) is divided by \( (x - 9) \).

iii) deduce the remainder when \( f(x^2 - 6) \) is divided by \( (x^2 - 8) \).

i) \( f(1) = 4(1)^3 - 2(1)^2 + 5(1) - 1 \)
\[ = 6 \]

ii) \( f(x - 8) = 4(x - 8)^3 - 2(x - 8)^2 + 5(x - 8) - 1 \)
When \( f(x - 8) \) is divided by \( (x - 9) \),
\[ f(9 - 8) = 4(9 - 8)^3 - 2(9 - 8)^2 + 5(9 - 8) - 1 \]
\[ f(1) = 4(1)^3 - 2(1)^2 + 5(1) - 1 \]
\[ = 6 \]

iii) \( f(x^2 - 6) = 4(x^2 - 6)^3 - 2(x^2 - 6)^2 + 5(x^2 - 6) - 1 \)
\[ f(8 - 6) = 4(8 - 6)^3 - 2(8 - 6)^2 + 5(8 - 6) - 1 \]
\[ f(2) = 4(2)^3 - 2(2)^2 + 5(2) - 1 \]
\[ = 33 \]

When the function \( f(x) \) is divided by \( (x + 1) \), the remainder is \(-5\). When \( f(x) \) is divided by \( (x - 1) \), the remainder is \(-1\). When \( f(x) \) is divided by \( (x^2 - 1) \), the remainder is \((Ax + B)\). Find \( A \) and \( B \).

\[ f(x) = (x^2 - 1)Q(x) + Ax + B \]
\[ f(x) = (x - 1)(x + 1)Q(x) + Ax + B \]
When \( f(x) \) is divided by \( (x + 1) \), remainder = \(-5\).
\[ f(-1) = -5 \]
\[-A + B = -5 \]

When \( f(x) \) is divided by \( (x - 1) \), remainder = \(-1\).
\[ f(1) = -1 \]
\[ A + B = -1 \]
Simultaneous solve (1) and (2), \( A = 2, B = -3 \)
$f(x)$ is a function where $f(x) = ax^3 + bx^2 + 2x - 5$. $2f(x) - 6$ is divisible by $(x - 1)$ and when $f(x) + 4$ is divided by $(x + 2)$, it leaves a remainder of $-5$. Find $A$ and $B$.

$2f(x) - 6 = 2ax^3 + 2bx^2 + 4x - 10 - 6$

Given that $2f(x) - 6$ is divisible by $(x - 1)$,

$2f(1) - 6 = 0$

$2a(1)^3 + 2b(1)^2 + 4(1) - 16 = 0$

$2a + 2b - 12 = 0$ ---(1)

Given that $f(x) + 4$ leaves a remainder of $-5$ when divided by $(x + 2)$,

$f(-2) + 4 = -5$

$a(-2)^3 + b(-2)^2 + 2(-2) - 5 + 4 = -5$

$-8a + 4b - 4 - 5 + 4 = -5$

$-8a + 4b = 0$ ---(2)

Simultaneously solve (1) and (2), $a = 2$ and $b = 4$. 

For Specialist Math Tuition/Coaching +65 90126407/ apexmathtuition@gmail.com
Unauthorized copying, resale or distribution prohibited Copyright © 2016 www.tuitionmath.com. All rights reserved
Given that \((x^2 - 3)\) is a factor of \(f(x) = x^3 + ax^2 + bx - 3\)

i) Find the value of \(a\) and \(b\).

ii) Hence, factorize \(f(x)\) completely.

iii) Hence, solve the equation \(1 + ay + by^2 - 3y^3 = 0\).

i) Let \(x^2 - 3 = 0\),
\[x = \pm \sqrt{3}\]

By factor theorem,
\[f(\sqrt{3}) = 0\] and \[f(-\sqrt{3}) = 0\].

\[
f(\sqrt{3}) = 0 \\
(\sqrt{3})^3 + a(\sqrt{3})^2 + b\sqrt{3} - 3 = 0 \\
3\sqrt{3} + 3a + b\sqrt{3} - 3 = 0 \quad \text{-(1)}
\]

\[
f(-\sqrt{3}) = 0 \\
(-\sqrt{3})^3 + a(-\sqrt{3})^2 + b(-\sqrt{3}) - 3 = 0 \\
-3\sqrt{3} + 3a - b\sqrt{3} - 3 = 0 \quad \text{-(2)}
\]

(1) + (2):
\[6a - 6 = 0\]
\[a = 1\]
Sub \(a = 1\) into (1): \(3\sqrt{3} + 3(1) + b\sqrt{3} - 3 = 0\)
\[b = -3\]

ii) \(x^3 + x^2 - 3x - 3 = (x^2 - 3)(Ax + B)\)
Comparing coefficient: \(A = 1, B = 1\)
\[
\therefore x^3 + x^2 - 3x - 3 = (x + \sqrt{3})(x - \sqrt{3})(x + 1)
\]

iii) \(1 + y - 3y^2 - 3y^3 = 0\)
\[
\frac{1}{y^3} + \frac{y}{y^3} - \frac{3y^2}{y^3} - \frac{3y^3}{y^3} = \frac{0}{y^3} \quad \text{(divide both sides by } y^3)\)
\[
(y^{-1})^3 + (y^{-1})^2 - 3(y^{-1}) - 3 = 0
\]
From part ii), \(x^3 + x^2 - 3x - 3 = (x + \sqrt{3})(x - \sqrt{3})(x + 1)\)
Sub \(x = y^{-1}\):
\[
(y^{-1})^3 + (y^{-1})^2 - 3(y^{-1}) - 3 = (y^{-1} + \sqrt{3})(y^{-1} - \sqrt{3})(y^{-1} + 1)
\]
Hence, \(y^{-1} = -\sqrt{3}\) or \(y^{-1} = \sqrt{3}\) or \(y^{-1} = -1\)
\[
\therefore y = -\frac{1}{\sqrt{3}} \text{ or } y = \frac{1}{\sqrt{3}} \text{ or } y = -1
\]