(1) Simultaneous, Quadrilaterals and Inequalities

1 Find the value(s) of \( k \) for the following simultaneous equations, given that the equations have no solution.

\[(k + 1)y = (2k - 1)x + 5 \quad \text{---(1)}\]
\[4y = (k + 2)x + 10 \quad \text{---(2)}\]

If both equations have no solution, the 2 lines are parallel and their \( y \)-intercepts are different.
Therefore, \( m_1 = m_2 \) and \( c_1 \neq c_2 \)

From (1) \[ y = \frac{2k-1}{k+1} x + \frac{5}{k+1} \]
From (2) \[ y = \frac{k+2}{4} x + \frac{10}{4} \]

\[ \frac{2k-1}{k+1} = \frac{k+2}{4} \]
\[ 8k - 4 = (k + 2)(k + 1) \]
\[ 8k - 4 = k^2 + 3k + 2 \]
\[ k^2 - 5k + 6 = 0 \]
\[ (k - 2)(k - 3) = 0 \]
\[ k = 2 \quad \text{or} \quad k = 3 \]

When \( k = 2 \):
Eqn (1): \[ 3y = 3x + 5 \]
\[ y = x + \frac{5}{3} \]
Eqn(2): \[ 4y = 4x + 10 \]
\[ y = x + \frac{10}{4} \]
Since \( c_1 \neq c_2 \), \( k = 2 \) is a valid solution.

When \( k = 3 \):
Eqn (1): \[ 4y = 5x + 5 \]
\[ y = \frac{5}{4} x + \frac{5}{4} \]
Eqn(2): \[ 4y = 5x + 10 \]
\[ y = \frac{5}{4} x + \frac{10}{4} \]
Since \( c_1 \neq c_2 \), \( k = 3 \) is a valid solution.

Therefore the values of \( k \) are 2 and 3.
2. The equation $2x^2 + 8x = 1$ has roots $\alpha$ and $\beta$.

a) State the value of $\alpha + \beta$ and $\alpha \beta$

b) Find the value of $\alpha^2 - \beta^2$, leaving your answer in surd form.

c) Find the quadratic equation whose roots are $\alpha^4 - \beta^4$

\[
\begin{align*}
\text{a)} & \quad 2x^2 + 8x - 1 = 0 \\
& \quad \alpha + \beta = -4 \\
& \quad \alpha \beta = -\frac{1}{2} \\
\text{b)} & \quad (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta \\
& \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \quad \text{(1)} \\
& \quad (\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta \\
& \quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta \quad \text{[Substitute (1)]} \\
& \quad (\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} \quad \text{(2)} \\
& \quad \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) \\
& \quad \alpha^2 - \beta^2 = (\alpha + \beta)\sqrt{(\alpha + \beta)^2 - 4\alpha\beta} \quad \text{[Substitute (2)]} \\
& \quad \alpha^2 - \beta^2 = (-4) \sqrt{(-4)^2 - 4 \left(-\frac{1}{2}\right)} \\
& \quad \alpha^2 - \beta^2 = -4\sqrt{18} \\
& \quad \alpha^2 - \beta^2 = -12\sqrt{2} \\
\text{c)} & \quad \alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 \\
& \quad = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2 \\
& \quad = \left[(-4)^2 - 2 \left(-\frac{1}{2}\right)\right]^2 - 2 \left(-\frac{1}{2}\right)^2 \\
& \quad = 288.5 \\
& \quad \alpha^4 \beta^4 = \left(-\frac{1}{2}\right)^4 = \frac{1}{16} \\
& \quad \text{Quadratic equation is: } x^2 - 288.5x + \frac{1}{16} = 0 \\
& \quad 16x^2 - 4616x + 1 = 0
\end{align*}
\]
It is given that $\alpha$ and $\beta$ are the roots of the equation $y = x^2 - x - 1$, where $\beta > \alpha$ and that $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$ are the roots of another quadratic equation with integer coefficients. Without solving the values of $\alpha$ and $\beta$, find the exact value of $\alpha + \frac{1}{\alpha}$.

\[
\begin{align*}
\alpha + \beta &= -\left(-\frac{1}{1}\right) = 1 \\
\alpha\beta &= -1 \\
\frac{1}{\alpha} + \beta + \frac{1}{\beta} &= \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta} \\
&= \alpha + \beta + \frac{\beta + \alpha}{\alpha\beta} \\
&= (1) + \frac{1}{-1} \\
&= 0
\end{align*}
\]

\[
\begin{align*}
\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right) &= \alpha\beta + \frac{\beta}{\alpha} + \frac{\alpha}{\beta} + \frac{1}{\alpha\beta} \\
&= \alpha\beta + \frac{\beta^2 + \alpha^2}{\alpha\beta} + \frac{1}{\alpha\beta} \\
&= \alpha\beta + \frac{(\beta + \alpha)^2 - 2\alpha\beta}{\alpha\beta} + \frac{1}{\alpha\beta} \\
&= (-1) + \frac{(1)^2 - 2(-1) + 1}{-1} \\
&= -1 - 3 - 1 \\
&= -5
\end{align*}
\]

\[
\therefore \text{ Equation with } \alpha + \frac{1}{\alpha} \text{ and } \beta + \frac{1}{\beta} \text{ as roots is: } y = x^2 - 5
\]

When $y = 0, x^2 = 5$

$x = \pm\sqrt{5}$

Since $\beta > \alpha$, $\alpha + \frac{1}{\alpha}$ is the smaller root.

\[
\therefore \alpha + \frac{1}{\alpha} = -\sqrt{5}
\]
4 a) If one root of the equation $4x^2 - 22x + k = 0$ is ten times the other, find the value of $k$.
b) Show that $2 - x^2 + 3x$ can never be greater than 5.

a) Let the roots be $\alpha$ and $10\alpha$,

Sum of roots $= -\frac{b}{a}$

$\alpha + 10\alpha = \frac{22}{4}$

$11\alpha = \frac{22}{4}$

$\alpha = \frac{2}{4} = 0.5$

Product of roots $= \frac{c}{a}$

$\alpha(10\alpha) = \frac{k}{4}$

$10\alpha^2 = \frac{k}{4}$

$10(0.5)^2 = \frac{k}{4}$

$k = 10$

b) $y = 2 - x^2 + 3x$ -(1)

$y = 5$ -(2)

Equate (1) with (2):

$2 - x^2 + 3x = 5$ 

$-x^2 + 3x - 3$

$b^2 - 4ac = 3^2 - 4(-1)(-3)$

$= -3$

Since $a < 0$, the graph is n-shaped

Since $b^2 - 4ac < 0$, the curve never touches the line $y = 5$

Hence, $2 - x^2 + 3x$ will never be greater than 5.
5. Show that the roots of the equation \( x^2 + (2 - k)x = \frac{3}{2}k \) are real for all real values of \( k \).

\[
x^2 + (2 - k)x - \frac{3}{2}k = 0 \quad \text{-(1)}
\]

To prove that the equation \( x^2 + (2 - k)x - \frac{3}{2}k = 0 \) has real roots for all real values of \( k \), we need to prove that the coefficient of \( x^2 \) term is positive (already proved by observation) and for Equation (1), the \( b^2 - 4ac > 0 \).

\[
b^2 - 4ac = (2 - k)^2 - 4 \left( \frac{3}{2}k \right) \\
= 4 + k^2 - 4k + 6k \\
= k^2 + 2k + 4 \quad \text{-(2)}
\]

To prove that \( k^2 + 2k + 4 > 0 \) for all values of \( k \), we need to prove that for Equation (2), \( b^2 - 4ac < 0 \).

Equation (2): \( b^2 - 4ac = 2^2 - 4(1)(4) = -12 < 0 \)

Since the coefficient of \( k^2 \) is positive and that \( b^2 - 4ac < 0 \) for Equation (2), \( k^2 + 2k + 4 > 0 \) for all values of \( k \).

Since for Equation (1), \( b^2 - 4ac > 0 \), it has real roots for all real values of \( k \).
The roots of the equation \( x^2 - 4x + k \) differs by \( 2s \). Show that \( s^2 = 4 - k \). Given also that the roots are positive integers and that \( k \) is a positive integer, find the possible values of \( s \).

Let the roots be \( \alpha, \alpha - 2s \).

Sum of roots \( = \frac{-b}{a} \)
\( \alpha + \alpha - 2s = 4 \)
\( \alpha = 2 + s \) \hspace{1cm} \text{(1)}

Product of roots \( = \frac{c}{a} \)
\( \alpha(\alpha - 2s) = k \) \hspace{1cm} \text{(2)}

Sub (1) into (2):
\( (2 + s)(2 + s - 2s) = k \)
\( (2 + s)(2 - s) = k \)
\( 4 - s^2 = k \)
\( s^2 = 4 - k \) \hspace{1cm} \text{(Shown)}

\( x^2 - 4x + k = 0 \)
\( x = \frac{4 \pm \sqrt{4^2 - 4k}}{2} \)
\( x = 2 \pm \sqrt{4 - k} \)

Given that roots are positive integers and \( k \) is a positive integer,

If \( k = 0 \), \( x = 4 \) or 0 (Rej since 0 is not positive integer)

If \( k = 1 \), \( x = 2 \pm \sqrt{3} \) (Rej since roots are not positive integer)

If \( k = 2 \), \( x = 2 \pm \sqrt{2} \) (Rej since roots are not positive integer)

If \( k = 3 \), \( x = 1 \) or 3

If \( k = 4 \), \( x = 2 \)

If \( k > 5 \), there will be no real roots

When \( k = 3 \), \( s = \pm 1 \)

When \( k = 4 \), \( s = 0 \)

Possible values of \( s \) are \(-1, 0 \) and \( 1 \).
7. Given that \( \alpha \) and \( \beta \) are the roots of the equation \( x^2 = x - 5 \), prove that

a) \( \frac{1 - \alpha}{5} = \frac{1}{\alpha} \)

b) \( \alpha^3 + 4\alpha + 5 = 0 \)

a)

Since \( \alpha \) is a root, \( \alpha^2 = \alpha - 5 \) \( -(1) \)

\( \alpha^2 - \alpha = -5 \)

\( \alpha(\alpha - 1) = -5 \)

\( \frac{\alpha - 1}{\alpha} = \frac{1}{\alpha - 5} \)

\( \frac{1 - \alpha}{5} = \frac{1}{\alpha} \) (Proved)

b) LHS = \( \alpha^3 + 4\alpha + 5 \)

= \( \alpha^2\alpha + 4\alpha + 5 \)

= \( (\alpha - 5)\alpha + 4\alpha + 5 \) (Substitute (1))

= \( \alpha^2 - 5\alpha + 4\alpha + 5 \)

= \( \alpha - 5 - \alpha + 5 \) (Substitute (1))

= 0

= RHS (Proved)
8  a) Find the range of values of $x$ for which $2x^2 + x - 6$ lies between $-3$ and $4$.
b) Show that if the roots of the equation $2x^2 + 3x - 2 + m(x - 1)^2 = 0$ are real, then $m$
cannot be greater than $\frac{25}{12}$.

a) $-3 < 2x^2 + x - 6 < 4$
   $-3 < 2x^2 + x - 6$ and $2x^2 + x - 3 > 0$
   $(x - 1)(2x + 3) > 0$

   $x < -1.5$ or $x > 1$

   $\therefore -2.5 < x < -1.5$ and $1 < x < 2$

b) $2x^2 + 3x - 2 + m(x - 1)^2 = 0$
   $2x^2 + 3x - 2 + mx^2 + m - 2xm = 0$
   $(2 + m)x^2 + (3 - 2m)x + m - 2 = 0$

   If roots are real, $b^2 - 4ac \geq 0$
   $(3 - 2m)^2 - 4(2 + m)(m - 2) \geq 0$
   $9 + 4m^2 - 12m - 8m - 16 \geq 0$
   $-12m + 25 \geq 0$
   $m \leq \frac{25}{12}$
Find the range of values of \( k \) for which the graph of \( y = kx^2 - 3x + kx \) lies entirely above the line \( y = 4 \).

\[
y = kx^2 + (k - 3)x \quad -(1)
\]
\[
y = 4 \quad -(2)
\]
Sub (1) into (2):
\[
kx^2 + (k - 3)x - 4 = 0
\]
If the graph lies entirely above line,
\[
b^2 - 4ac < 0 \quad \text{and} \quad a > 0
\]
\[
(k - 3)^2 - 4(4)(-4) < 0 \quad \text{and} \quad k > 0 \quad (since \ a = k)
\]
\[
k^2 + 9 - 6k + 16k < 0 \quad \text{and} \quad k > 0
\]
\[
(k + 1)(k + 9) < 0 \quad \text{and} \quad k > 0
\]

\[
-9 < k < -1 \quad \text{and} \quad k > 0
\]
Therefore, there are no real values of \( k \) for which the graph lies entirely above the line.
10  

i) Show that the expression $x^2 - x + \frac{7}{2}$ is always positive for all real values of $x$.

ii) Hence, find the values of $k$ which satisfy the inequality $\frac{-x^2+2+kx+2}{-(x^2-x+3.5)} < 2$ for all real values of $x$.

i) $x^2 - x + \frac{7}{2}$

\[ b^2 - 4ac = (-1)^2 - 4(1)(\frac{7}{2}) \]
\[ = -13 \]

Since $a > 0$ and $b^2 - 4ac < 0$, the curve is always positive.

ii) $\frac{-x^2+kx+2}{-(x^2-x+3.5)} < 2$

\[-x^2 + kx + 2 > -2x^2 + 2x - 7 \]
\[ x^2 + (k - 2)x + 9 > 0 \]

\[ b^2 - 4ac < 0 \]
\[ (k - 2)^2 - 4(1)(9) < 0 \]
\[ k^2 - 4k - 36 < 0 \]
\[ k^2 - 4k - 32 < 0 \]
\[ (k - 8)(k + 4) < 0 \]

\[ \therefore -4 < k < 8 \]
11 The roots of the equation $2x^2 - 8x + 50 = 0$ are $\alpha^2$ and $\beta^2$. Find
i) the value of $\alpha^2 + \beta^2$ and $\alpha^2\beta^2$.
ii) two different quadratic equations whose roots are $\alpha$ and $\beta$

i) $\alpha^2 + \beta^2 = 4$
$\alpha^2\beta^2 = 25$

ii) $\alpha\beta = \sqrt{25} = 5$
$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$
$\alpha + \beta = \pm\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta}$
$\alpha + \beta = \pm\sqrt{4 + 2(5)}$
$= \pm\sqrt{14}$

$x^2 + \sqrt{14} + 5 = 0$
$x^2 - \sqrt{14} + 5 = 0$